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A GENERALIZATION OF RAMSEY THEORY FOR GRAPHS--WITH STARS AND COMPLETE GRAPHS AS FORBIDDEN SUBGRAPHS

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I. INTRODUCTION

In [1], the following generalization of Ramsey Theory for graphs was introduced: Let c and d be two integers such that $c > d \geq 1$. Let t denote the binomial coefficient $\binom{c}{d}$. Given c distinct colors, we order the t subsets of d colors in some arbitrary manner. Let $G_1, G_2, \ldots, G_t$ be graphs. The d-chromatic Ramsey number, denote by $r_{d}^{c}(G_1, G_2, \ldots, G_t)$, is defined to be the least number p such that if the edges of the complete graph K_p are colored in any fashion with c colors; then for some i the subgraph whose edges are colored with i^{th} subset of colors contains a G_i. In [1], the case $c = 3$, $d = 2$, G_1, G_2, G_3 being complete graphs was studied. In this paper we further study cases where G_1, G_2, G_3 are stars and complete graphs. Thus, for instance, $r_2^3(K_{1,i}, K_{1,j}, K_{1,\ell})$ denotes the smallest p such that if the edges of the complete graph K_p are colored with the colors α, β, and γ, there is either an $\alpha - \beta$ $K_{1,i}$ or an $\alpha - \gamma$ $K_{1,j}$ or a $\beta - \gamma$ $K_{1,\ell}$ in the graph K_p.

$$r_2^3(K_{1,i}, K_{1,j}, K_{1,\ell}) = \begin{cases} i + j - 1 & \ell \geq i + j;\ i,j \text{ both even} \\ i + j & \ell \geq i + j;\ i,j \text{ not both even} \\ \lceil \frac{1}{2}(i+j+\ell) \rceil - 1 & \ell < i + j;\ (i,j,\ell) \in S \\ \lceil \frac{1}{2}(i+j+\ell) \rceil & \ell < i + j;\ (i,j,\ell) \notin S \end{cases}$$

where $S = \{(i,j,\ell) \mid i + j + \ell \equiv 3 \bmod 4$ and not all i, j, ℓ are odd$\}$

Proof: For the cases $\ell \geq i + j$, we have [2]:

$$r(K_{1,i}, K_{1,j}) = \begin{cases} i + j - 1 & \text{if } i,j \text{ are both even} \\ i + j & \text{otherwise} \end{cases}$$

Therefore by Theorem 2.2:

$$r_2^3(K_{1,i}, K_{1,j}, K_{1,\ell}) = \begin{cases} i + j - 1 & \text{if } \ell \geq i + j - 1, \text{ and } i,j \text{ are both even} \\ i + j & \text{if } \ell \geq i + j,\ i,j \text{ are not both even} \end{cases}$$

For cases $\ell < i + j$, we establish lower bounds by giving a critical coloring for each of case when $(k,j,\ell) \in S$ and $(i,j,\ell) \notin S$.

For the case of $i + j + \ell \equiv 3 \bmod 4$, we consider the coloring of K_p, $p = \frac{1}{2}(i+j+\ell-3)$. Since p is even, according to Lemma 3.2, we can decompose K_p into regular graphs G_α, G_β, and G_γ, whose degrees are $\frac{1}{2}(i+j-\ell-3)$, $\frac{1}{2}(i-j+\ell-1)$, and $\frac{1}{2}(-i+j+\ell-1)$, respectively. We color G_α with color α, G_β with color β, and G_γ with color γ. It is easy to check that $N_\alpha + N_\beta < i$, $N_\alpha + N_\gamma < j$ and $N_\beta + N_\gamma < \ell$ at each vertex.

For the case of $(i,j,\ell) \in S$, we consider the coloring of K_p, $p = \lceil \frac{1}{2}(i+j+\ell) \rceil - 1$. There are 4 subcases:

Case 1: If $i + j + \ell \equiv 0 \bmod 4$, $p = \frac{1}{2}(i+j+\ell) - 1$ and p is odd. Let $a = \frac{1}{2}(i+j-\ell) - 1$, $b = \frac{1}{2}(i-j+\ell) - 1$, $c = \frac{1}{2}(-i+j+\ell) - 1$. Since $a + b + c = \frac{1}{2}(i+j+\ell) - 3$ which is odd, either one or all of a, b, and c are odd. There are 2 possibilities.

Suppose a is odd, but b and c are even. We decompose K_p into regular graphs G_α, G_β, G_γ of even degrees $N_\alpha = a + 1$, $N_\beta = b$, and $N_\gamma = c$. We color G_α with color α, G_β with color β, and G_γ with color γ. Similar coloring can be made if either b or c is odd.

Suppose a, b, and c are all odd. We first consider the coloring of a complete subgraph K_{p-1} of K_p. We decompose K_{p-1} into regular graph G_α, G_β and G_γ of degree a, b, and c respectively. G_α is then colored with color α, G_β with β, and G_γ with γ. We color the p - 1 edges that connect the remaining vertex to the other vertices in the following fashion: a + 1 edges are colored with α, b edges with β, and c edges with r. Since at each vertex of K_p, either $N_\alpha = a + 1$, $N_\beta = b$, $N_\gamma = c$, or $N_\alpha = a$, $N_\beta = b + 1$, $N_\gamma = c$, or $N_\alpha = a$, $N_\beta = b$, $N_\gamma = c + 1$, it can be easily checked that there is indeed no $\alpha - \beta$ $K_{1,i}$, nor an $\alpha - \gamma$ $K_{1,j}$, nor a $\beta - \gamma$ $K_{1,\ell}$.

Case 2: Suppose $i + j + \ell \equiv 1 \bmod 4$, $p = \lceil \frac{1}{2}(i+j+\ell) \rceil = \frac{1}{2}(i+j+\ell-1)$ and is even. We decompose K_p into regular graphs G_α, G_β, and G_γ whose degrees are $\frac{1}{2}(i+j-\ell-1)$, $\frac{1}{2}(i-j+\ell-1)$ and $\frac{1}{2}(-i+j+\ell-1)$ and which are then colored with colors α, β, and γ, respectively.

Case 3: If $i + j + \ell \equiv 2 \bmod 4$, $p = \frac{1}{2}(i+j+\ell) - 1$ and is even. We decompose K_p into regular graphs G_α, G_β, and G_γ whose degrees are $\frac{1}{2}(i+j-\ell)$, $\frac{1}{2}(i-j+\ell-1)$ and $\frac{1}{2}(-i+j+\ell-1)$ and which are then colored with colors α, β, and γ $\frac{1}{2}(-i+j+\ell-1)$, respectively.

Case 4: If $i + j + \ell \equiv 3 \bmod 4$ and i,j,ℓ all odd, $p = \frac{1}{2}(i+j+\ell) - 1$ and is odd. We decompose K_p into regular graphs G_α, G_β, and G_γ, whose degrees are $\frac{1}{2}(i+j-\ell-1)$, $\frac{1}{2}(i-j+\ell-1)$, and $\frac{1}{2}(-i+j+\ell-1)$, all even, and which are then colored with colors α, β, and γ, respectively.

We next show that for $(i,j,\ell) \in S$, it is not possible to color a K_p, $p = \lceil \frac{1}{2}(i+j+\ell) \rceil - 1$ that does not contain an $\alpha - \beta$ $K_{1,i}$, or an $\alpha - \gamma$ $K_{1,j}$, or a $\beta - \gamma$ $K_{1,\ell}$. From (3.1), (3.2), and (3.3), since $N_\alpha + N_\beta + N_\gamma = p - 1 = \frac{1}{2}(i+j+\ell+1) - 2$, we must have $N_\gamma \geq \frac{1}{2}(-i+j+\ell-1)$, $N_\beta \geq \frac{1}{2}(i-j+\ell-1)$ and $N_\alpha \geq \frac{1}{2}(i+j-\ell-1)$, at each vertex. But since $N_\alpha + N_\beta + N_\gamma = \frac{1}{2}(i+j+\ell+1) - 2$, we must have at each vertex, $N_\alpha = \frac{1}{2}(+i+j-\ell-1)$, $N_\beta = \frac{1}{2}(i-j+\ell-1)$, $N_\gamma = \frac{1}{2}(-i+j+\ell-1)$. If $i = 2n_1 + 1$, $j = 2n_2$, $\ell = 2n_3$ then $i + j + \ell = 3 \bmod 4$ implies $(n_1+n_2+n_3)$ is odd. Therefore N_α is also odd. Since p is odd, then there is no coloring for G_α.

We still have to consider the case when $\ell = i + j$ and both i and j are even. Consider coloring a K_p, $p = i + j - 1$. Applying the above argument, we must have at each vertex $N_\alpha = 0$, $N_\beta = \frac{1}{2}(i-j+\ell) - 1 = i - 1$, and $N_\gamma = \frac{1}{2}(-i+j+\ell) - 1 = j - 1$. But again since p and N_α are both odd, no coloring is possible. ☐

IV. RAMSEY NUMBERS FOR STARS AND COMPLETE GRAPHS

Results have been obtained for evaluating exact values of the Ramsey numbers $r_2^3(K_{1,i}, K_j, K_\ell)$, for small i and j. Since the proofs are lengthy and involved, they will not be included. See [3] for details. Without lost of generality, we assume that $j \leq \ell$. The results are:

Theorem 4.1:

$$r_2^3(K_{1,2}, K_j, K_\ell) = r_2^3(P_3, K_j, K_\ell) = \begin{cases} 2j - 1 & \ell \geq 2j - 1 \\ \frac{2}{3}(j+\ell-1) & \ell < 2j - 1 \text{ and } j + \ell \equiv 1 \pmod 3 \\ 2\left\lceil \frac{j+\ell}{3} \right\rceil - 1 & \text{otherwise} \end{cases}$$

Theorem 4.2:

$$r_2^3(K_{1,3}, K_j, K_j) = 2j - 1$$

Theorem 4.3:

$$r_2^3(K_{1,i}, K_3, K_\ell) \begin{cases} = 2i + 1 & \ell \geq 2i + 1 \\ = 2i & \ell = 2i \\ = 2i - 1 & i \leq \ell \leq 2i - 1 \\ \leq \ell + i - 2 & \ell < i,\ i \text{ even, } \ell \text{ odd} \\ \leq \ell + i - 1 & \text{otherwise} \end{cases}$$

Corollary 4.3.1:

$$r_2^3(K_{1,4}, K_3, K_\ell) = \begin{cases} 5 & \ell = 3 \\ 7 & 4 \leq \ell \leq 7 \\ 8 & \ell = 8 \\ 9 & \ell \geq 9 \end{cases}$$

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. Abstracts

Let c and d be two integers such that $c \geq d \geq 1$. Let t denote the binomial coefficient $\binom{c}{d}$. Given c distinct colors, we order the t subsets of d colors in some arbitrary manner. Let $G_1, G_2, \ldots, G_t$ be graphs. The d-chromatic Ramsey number, denoted by $r_d^c(G_1, G_2, \ldots, G_t)$ is defined to be the least number p such that if the edges of complete graph K_p are colored in any fashion with c colors, then for some i the subgraph whose edges are colored with the i^{th} subset of colors contains a G_i. In this report, we study the case $c = 3$, $d = 2$. Results are obtained for cases where G_1, G_2, G_3 are complete graphs or stars.

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